

Solvability of the cohomological equation for regular vector fields on the plane

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Abstract

We consider planar vector field without zeroes ξ and study the image of the associated Lie derivative operator L_ξ acting on the space of smooth functions. We show that the cokernel of L_ξ is infinite-dimensional as soon as ξ is not topologically conjugate to a constant vector field and that, if the topology of the integral trajectories of ξ is “simple enough” (e.g. if ξ is polynomial) then ξ is transversal to a Hamiltonian foliation. We use this fact to find a large explicit subalgebra of the image of L_ξ and to build an embedding of \mathbb{R}^2 into \mathbb{R}^4 which rectifies ξ . Finally we use this embedding to characterize the functions in the image of L_ξ .

1 Introduction, definitions and main results

The study of planar vector fields has a long history going back to Poincaré and Bendixson (see [God83] and [CC03] for more details and bibliography on this topic). The topological classification of *regular* (i.e. without zeros) vector fields on the plane was completed first by W. Kaplan [Kap40, Kap48], using an ad-hoc topological tool (chordal systems), based on previous works of his advisor H. Whitney [Whi33b, Whi33a, Whi41]. In this paper we rather use the more general concept of *inseparable leaves* and *separatrices*, introduced by L. Markus [Mar54] while working at the extension of Kaplan’s results to the more general problem of the topological classification of all planar vector fields.

We recall a few standard basic concepts and definitions that will be used in the paper. We denote by $\mathfrak{X}_r(\mathbb{R}^2)$ the set of all smooth regular vector fields in the plane, by \mathcal{F}_ξ the foliation of the integral trajectories of $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$ and

by $\pi_\xi : \mathbb{R}^2 \rightarrow \mathcal{F}_\xi$ the canonical projection that sends every point in the leaf¹ passing through it. We endow \mathcal{F}_ξ with the canonical quotient topology. It was shown by Haefliger and Reeb [HR57] that \mathcal{F}_ξ admits the structure of a 1-dimensional simply connected second countable non (necessarily) Hausdorff smooth manifold; the smooth structure is characterized by the property that the restriction of π_ξ to every transversal line ℓ is a diffeomorphism onto its image. Two integral trajectories s_i , $i = 1, 2$, of ξ are said *inseparable* when their projections $\pi_\xi(s_i)$ cannot be separated in the topology of \mathcal{F}_ξ (e.g. see Fig. 1). We denote by $\mathcal{I}_{\mathcal{F}_\xi, s}$ the set of all leaves distinct from s inseparable from it (note that $\mathcal{I}_{\mathcal{F}_\xi, s}$ is empty for all but countably many leaves) and by $\mathcal{S}_{\mathcal{F}_\xi}$ the (countable) set of leaves for which $\mathcal{I}_{\mathcal{F}_\xi, s}$ is not empty. A leaf s is called a *separatrix* when the boundary of every neighbourhood of $\pi_\xi(s)$ contains more than two points. The set of all separatrices is the closure of $\mathcal{S}_{\mathcal{F}_\xi}$ [Mar54]. In the present paper we will rather use the term separatrix to indicate just the elements of $\mathcal{S}_{\mathcal{F}_\xi}$ since their limit points play no role in our work. Every plane foliation is orientable and, correspondingly, to each set $\mathcal{I}_{\mathcal{F}_\xi, s}$ can be given a natural order; we say that two separatrices are *adjacent* if they are next to each other with respect to this order.

We introduce now a few specific definition we will need throughout the paper.

Definition 1. *Two vector fields ξ and ξ' are strongly proportional if they are proportional through a non-zero smooth function. A vector field ξ is intrinsically Hamiltonian if it is strongly proportional to a Hamiltonian vector field and is transversally Hamiltonian if it is transversal to a Hamiltonian foliation \mathcal{G} , i.e. to the level sets of a regular smooth function G (we say that G is a Hamiltonian for G).*

It is easily seen that a regular vector field is intrinsically Hamiltonian iff the PDE $L_\xi f = 0$ admits a regular smooth solution and is transversally Hamiltonian iff it is solvable the differential inequality $L_\xi f > 0$.

Definition 2. *A foliation \mathcal{F}_ξ (or simply the vector field ξ) is of finite type if $\mathcal{S}_{\mathcal{F}_\xi}$ is closed and every set $\mathcal{I}_{\mathcal{F}_\xi, s}$ is finite.*

In this case the complement of the set of separatrices is the disjoint union of countably many unbounded open sets named by Markus [Mar54] *canonical regions* and the boundary of each canonical region has a finite number of connected components. We recall that examples of smooth or even analytic foliations of the plane with a dense set of separatrices are known in literature (see [Waz34] and [Mul76a]). While there are reasons to believe that such foliations are generic in some “combinatorial” sense, the set of foliations of finite type is nevertheless of great importance since important natural categories of regular vector fields leads to them. For example every polynomial vector field is of finite type: finite bounds for the number of the inseparable leaves of a polynomial vector field were first found by Markus [Mar72] and later improved

¹Throughout the paper we refer to the points of \mathcal{F}_ξ as *integral trajectories* or *leaves* depending on the aspect of them we want to emphasize.

independently by M.P. Muller [Mul76b] and S. Schecter and M.F. Singer [SS80]. It is easy to verify that are of finite type also all regular vector fields strongly proportional to those of the kind $\xi(x, y) = (a(y), b(y))$, where (a, b) is a generic pair of Morse functions of one variable (so that $a^2 + b^2$ is strictly positive).

Definition 3. A complete set of transversals (CST) for \mathcal{F}_ξ is a set of lines $\mathcal{T}_\xi = \{\ell_i\}$, one for each separatrix of \mathcal{F}_ξ , such that every ℓ_i is transversal to \mathcal{F}_ξ and cuts the corresponding separatrix s_i and the set $\{\pi_\xi(\ell_i)\}$ covers \mathcal{F}_ξ .

We call gap of $g \in C^\infty(\mathbb{R}^2)$ between two adjacent separatrices s_1 and s_2 with respect to the CST \mathcal{T}_ξ the limit (if it exists)

$$\text{gap}(g; s_1, s_2) = \lim_{\substack{p \rightarrow p_1 \\ \mathcal{T}_\xi}} \int_0^{T_p} g(\Phi_\xi^t(p)) dt,$$

where the point $p \in \ell_1$ tends to $p_1 = \ell_1 \cap s_1$, Φ_ξ^t is the flux of ξ and T_p is the unique number s.t. $\Phi_\xi^{T_p}(p) \in \ell_2$ ²

Finally we set a few notations on spaces of germs we are going to use in the last section. Let $a \in \mathbb{R}$. We denote by H_a^r the ring of left germs at a of functions in $C^r(-\infty, a)$, i.e. the equivalence classes determined by the equivalence relation $h \simeq h'$ if h and h' coincide in some interval of the form $(a - \epsilon, a)$ for some $\epsilon > 0$, and by G_a^r the subring of the left germs in H_a^r which can be extended to a continuous function at a together with their derivatives up to order r . Similarly, let $I = \{a\} \times [b_1, b_2]$ and set $L_I = (-\infty, a] \times \mathbb{R} \setminus I$. We denote by H_I^r the ring of left germs at I of functions of $C^r(L_I)$, i.e. $h \simeq h'$ if h and h' coincide in some set $(U \cap L_I) \setminus I$, where U is a neighbourhood of I , and by G_I^r the subring of germs of functions of H_I^r which can be extended to C^r functions on the whole L_I .

Definition 4. We call singular left germs at $a \in \mathbb{R}$ the elements of the quotient ring $SG_a^r = H_a^r/G_a^r$ and singular left germs at $I = \{a\} \times [b_1, b_2]$ the elements of the quotient ring $SG_I^r = H_I^r/G_I^r$.

Let us now turn to the topics of the present paper. Recently S.P. Novikov [Nov08] (in case of smooth functions) and G. Forni [For97] (for functional spaces of integrable functions) proved, as a generalization of the well-known Diophantine phenomena in the torus, that the first order homogeneous partial differential operator associated to a generic vector field on a compact surface has an infinite-dimensional cokernel. In Section 2 we generalize this result to the plane with the following result:

Theorem 1. Either ξ is topologically conjugated to the constant vector field, in which case $\dim \text{coker } L_\xi = 0$, or $\dim \text{coker } L_\xi = \infty$.

It was proved by J. Weiner [Wei88], using a different terminology, that every Hamiltonian foliation is transversally Hamiltonian. In Section 3 we extend Weiner's result in the following way:

²Such number exists for s_1 and s_2 are inseparable and is unique for every transversal cuts each leaf at most once.

Theorem 2. *Every plane foliation of finite type is transversally Hamiltonian.*

In Section 4 we use Theorem 2 to characterize the image of L_ξ when ξ is intrinsically Hamiltonian or of finite type. We recall that the question of the solvability of the so-called cohomological equation

$$L_\xi f = g, \quad (1)$$

is of purely global nature: it is well known indeed that, if ξ is regular, every point $p \in \mathbb{R}^2$ has a neighbourhood U_p such that $L_\xi(C^\infty(U_p)) = C^\infty(U_p)$. A solution to (1) in U_p is given explicitly by

$$f(p) = \int_0^{T_p} g(\Phi_\xi^t(p_\ell)) dt + h(\varphi(p)),$$

where Φ_ξ^t is the flow of ξ , T_p is the time needed to reach p under the action of Φ_ξ^t from the point p_ℓ lying on a fixed line ℓ embedded in U_p and everywhere transversal to ξ , $\varphi \in C^\infty(U_p)$ is the (functional) generator of $\ker L_\xi$ and $h \in C^\infty(\mathbb{R})$. The most general global result known for the action of a single regular vector field on the space of smooth functions is the following theorem by Hörmander and Duistermaat [DH72], which shows that a non-trivial behaviour of L_ξ must correspond to a non-trivial structure of the foliation \mathcal{F}_ξ of the integral trajectories of ξ and viceversa:

Theorem DH. *Let M be an open connected manifold and $\xi \in \mathfrak{X}_r(M)$. Then $L_\xi(C^\infty(M)) = C^\infty(M)$ iff \mathcal{F}_ξ admits a global transversal³.*

For the case of regular vector fields in \mathbb{R}^2 we show (see Proposition 4 and the paragraph below it) that for every ξ intrinsically Hamiltonian or of finite type there exist two commuting vector fields ξ'_F and ξ'_G , the first strongly proportional to ξ and the second transversal to it (ξ'_G diverges on some separatrices in the finite type case but this does not hinder the result), for which the following holds:

Theorem 3. *A function $g \in C^\infty(\mathbb{R}^2)$ belongs to $L_{\xi'_F}(C^\infty(\mathbb{R}^2))$ iff all functions $L_{\xi'_G}^k g$, $k \in \mathbb{N}$, have finite gap for all pairs of adjacent separatrices of ξ .*

Finally, in Section 5 we show that there exists an embedding $\hat{\Phi}_{FG}$ of \mathbb{R}^2 into \mathbb{R}^4 which rectifies globally both ξ'_F and ξ'_G at the same time. This setting for the cohomological equation is dual to the initial one in the following sense: in the original one the ambient space is always the same and the qualitative difference between inequivalent cohomological equations resides in the analytic expression of the vector field; in the embedding, instead, the cohomological equation has always the same analytic expression and is the geometry of the ambient space $\hat{\Phi}_{FG}(\mathbb{R}^2)$ that determines the solvability of the equation.

In this second setting we prove the following results:

³By global transversal we mean a codimension-1 embedded submanifold of M which is transversal to \mathcal{F}_ξ at every point and cuts every leaf exactly once.

Theorem 4. *There exists a countable family of intervals $I_j = \{a_j\} \times [b_{j,1}, b_{j,2}]$ and of ring homomorphisms $\theta_j^{(r)} : SG_{I_j}^r \rightarrow SG_{a_j}^r$ such that $g \in L_{\xi'}(C^r(\mathbb{R}^2))$ iff $[(\hat{\Phi}_{FG})_* g]_{SG_{I_j}} \in \ker \theta_j^{(r)}$ for all $\theta_j^{(r)}$.*

Note that in this paper we are interested only to the action of ξ on smooth functions; concerning the global solvability in other functional spaces, e.g. of entire functions or Gevrey-type functions in the realm of global Cauchy-Kowalevskaya theorem see [GM03, GG07] and the references therein. Note also that there is a qualitative difference between the case of a single operator L_ξ and the case of two or more operators $\{L_{\xi_1}, \dots, L_{\xi_k}\}$; it has been shown indeed by M. Gromov [Gro86] that, on every smooth manifold M ,

$$L_{\xi_1}(C^\infty(M)) + L_{\xi_2}(C^\infty(M)) = C^\infty(M)$$

for any pair of vector fields in mutual generic position.

2 coker L_ξ

As pointed out above in Theorem DH, if \mathcal{F}_ξ admits a global transversal the method of characteristics provides a global solution to the cohomological equation (1) for every $g \in C^\infty(\mathbb{R}^2)$, so that $L_\xi(C^\infty(\mathbb{R}^2)) = C^\infty(\mathbb{R}^2)$ and $\text{coker } L_\xi = \{0\}$. The obstruction to the existence of global transversals is the presence of separatrices since no smooth line ℓ can, at the same time, be transversal to \mathcal{F}_ξ and intersect any pair leaves inseparable from each other.

In absence of global transversals, one can try to solve $L_\xi f = g$ recursively in the following way. Let s be a separatrix for ξ and denote by ℓ any transversal through it and by $U_\ell = \pi_\xi^{-1}(\ell) \subset \mathbb{R}^2$ the saturated open set containing ℓ . Since U_ℓ is a proper subset of \mathbb{R}^2 , its boundary is non-empty and equal to the union of the sets $\mathcal{I}_{\mathcal{F}_\xi, \tilde{s}}$ corresponding to all leaves \tilde{s} cut by ℓ . By construction ξ , once restricted to U_ℓ , admits a global transversal (the line ℓ) and therefore $L_\xi(C^\infty(U_\ell)) = C^\infty(U_\ell)$. Let now g_ℓ be any solution, in U_ℓ , to $L_\xi f = g$. We can try to extend g_ℓ beyond U_ℓ by selecting any boundary component s' of ∂U_ℓ and any transversal ℓ' passing through it. The function g_ℓ restricts on $\ell' \cap U_\ell$ to a smooth function $\hat{g}_{\ell'}$; if we can extend $\hat{g}_{\ell'}$ to a smooth function $g_{\ell'}$ defined on the whole ℓ' then, via the method of characteristics applied to the set $U_{\ell'} = \pi_\xi^{-1}(\ell')$ and using $g_{\ell'}$ as initial condition on ℓ' , we can smoothly extend g_ℓ to $U_{\ell'}$. Assuming that one can always extend a local solution across transversals as described above, proceeding recursively until no separatrices are left we end up with a global solution to (1).

We are going to use the gap to provide a quantitative criterion for the existence of continuous solutions. While the gap of a function clearly depends on the CST chosen, whether it exists and is bounded does not:

Proposition 1. *If the gap of $g \in C^\infty(\mathbb{R}^2)$ between two adjacent separatrices s_1 and s_2 with respect to a CST \mathcal{T}_ξ exists and it is finite, then it exists and it is finite also with respect to every other CST \mathcal{T}'_ξ .*

Proof. Let $\ell'_1, \ell'_2 \in \mathcal{T}'_\xi$ be the two transversal to s_1 and s_2 in the second CST. Then

$$\text{gap}_{\mathcal{T}'_\xi}(g; s_1, s_2) = \text{gap}_{\mathcal{T}_\xi}(g; s_1, s_2) + A_1 + A_2$$

for

$$A_1 = \int_{p'_1}^{p_1} g(\Phi_\xi^t(p'_1)) dt, \quad A_2 = \int_{p_2}^{p'_2} g(\Phi_\xi^t(p_2)) dt$$

where the integral defining A_i , $i = 1, 2$, is evaluated along s_i . Recall that, due to the method of characteristics, the values on a leaf of a local solution to the cohomological equation are completely determined by the value of the solution in any point of the leaf and they are finite on the whole leaf iff they are finite at a single point. Hence, if the gap of g between s_1 and s_2 with respect to \mathcal{T}_ξ , both A_i are finite since they are given by integrals of bounded functions over compact sets. \square

It is already implicit in the previous proof that the existence and boundedness of the gap of a function g is related to the extendability of local solutions of the cohomological equation having g as rhs. Below we prove this fact and then use it to prove the main result of the section.

Proposition 2. *A global continuous solution to $L_\xi f = g$ exists iff g has finite gap between every pair of adjacent separatrices of \mathcal{F}_ξ .*

Proof. We point out first that a continuous solution to $L_\xi f = g$, $g \in C^\infty(\mathbb{R}^2)$, is much more regular than it sounds since all such solutions are, by definition, smooth in the ξ direction. In particular the integral of df along the integral trajectories of ξ is well-defined even for continuous solutions of (1) since the restriction of df on these integral trajectories depends only on $L_\xi f$.

The condition in the hypothesis of the theorem is clearly necessary for, if a continuous solution f exists, then for a given \mathcal{T}_ξ we have

$$\text{gap}_{\mathcal{T}_\xi}(g; s_1, s_2) = \lim_{p \rightarrow p_1} \int_0^{T_p} df = f(p_2) - f(p_1).$$

Note that the gap of g between s_1 and s_2 depends only on the intersection of the two separatrices with the relative transversals in \mathcal{T}_ξ .

Now assume that a solution f_1 is defined in $U_1 = \pi^{-1}(\ell_1)$ and that the gap of g between s_1 and s_2 is finite. Then the restriction of f_1 on ℓ_2 can be continued to a continuous function on the whole ℓ_2 and therefore, via the method of characteristics, to the whole $U_2 = \pi^{-1}(\ell_2)$. The new function f_2 defined on $U_1 \cup U_2$ coincides, by construction, with f_1 in $U_1 \cap U_2$, is continuous in $U_1 \cup U_2$ and clearly does not depend on the choice of the particular CST used in the extension. By proceeding recursively until all separatrices are taken into account we end up with a global continuous solution to (1). \square

We are now in condition to prove Theorem 1.

Theorem 1. *If ξ has at least a pair of separatrices then $\dim \operatorname{coker} L_\xi = \infty$.*

Proof. We can assume without loss of generality that ξ is complete⁴. Under this assumption the gap of every non-zero constant function is infinite for it is proportional to T_p , which clearly diverges for $p \rightarrow p_1$. Then the gap diverges also on every function which is minored by a non-zero constant, e.g. the polynomials $p_{n,m}(x, y) = 1 + x^{2n} + y^{2m}$, so that the image of L_ξ misses infinitely many linearly independent functions, i.e. $\dim \operatorname{coker} L_\xi = \infty$. \square

3 $L_\xi f > 0$

Finding criteria to characterize functions belonging to the image of L_ξ is hard and in the case of a generic regular vector field we cannot state much more than the fact that a necessary condition (but far from being sufficient) to belong to it is to have finite gap between all pairs of adjacent separatrices. More can be said for the vector fields which are transversally Hamiltonian, which makes crucial studying the solvability of the differential inequality $L_\xi f > 0$.

Proposition 3. *Let $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$, $\Omega_0 = dx \wedge dy$ and $\omega_\xi = i_\xi \Omega_0$. The following conditions are equivalent:*

1. \mathcal{F}_ξ is transversally Hamiltonian;
2. the inequality $L_\xi f > 0$ has a smooth solution;
3. $\omega_\xi \wedge df$ is a volume form for some $f \in C^\infty(\mathbb{R}^2)$.

Proof. Let \mathcal{G} be a Hamiltonian foliation transversal to \mathcal{F}_ξ and G a Hamiltonian for \mathcal{G} . Since $T\mathcal{G} = \ker dG$ we must have $dG(\xi) \neq 0$ at every point, so that either $L_\xi G > 0$ or $L_\xi(-G) > 0$ and viceversa. Part 3 is due to the fact that $\omega_\xi \wedge dG = i_\xi dG \Omega_0 = L_\xi G \Omega_0$. \square

As mentioned in the introduction, Weiner [Wei88] proved that every intrinsically Hamiltonian falls in this class. Below, after a preparatory Lemma, we extend this result to every ξ of finite type.

Lemma 1. *Let ξ be a regular vector field of finite type. Then \mathcal{F}_ξ admits a CST with the following property: for each separatrix $s \in \mathcal{S}$, the saturated open set $\pi_\xi^{-1}(\pi_\xi(\ell))$ of all leaves cutting the corresponding transversal $\ell \in \mathcal{T}$ is equal to the union of s with the two canonical regions having s as boundary component.*

Proof. Let s be a separatrix, U one of the two canonical regions having s as boundary, ℓ the corresponding transversal in \mathcal{T} and ℓ_U the connected component of $\ell \setminus s$ which intersects U . Since U admits a global transversal, there is a natural diffeomorphism ψ of U into \mathbb{R} sending the leaves of \mathcal{F}_ξ into vertical lines. If $\pi_\xi^{-1}(\pi_\xi(\ell_U)) \neq U$ there is no geometrical obstruction to make $\psi(\ell_U)$ either shorter or longer in the horizontal direction while keeping it transversal

⁴This is true for any smooth vector field on a manifold, e.g. see [God83], Proposition 1.19; in this case, since ξ is regular, we could simply assume that it has unitary Euclidean length.

to the vertical direction and without modifying it close to s so that the first projection of $\psi(\ell)$ on the first factor is surjective. After we do the same on the second canonical region V we are left with a new transversal ℓ' such that $\pi_\xi^{-1}(\pi_\xi(\ell')) = U \cup V \cup s$. \square

Theorem 2. *Every regular vector field of finite type is transversally Hamiltonian.*

Proof. We can assume without loss of generality that ξ is complete and denote by \mathcal{T}_ξ any CST having the property described in the Lemma above. The collection of open subsets $V_{s,i}$ defined by

$$V_{s,i} = \{\Phi_\xi^t(\ell_s) \mid t \in (i, i+1)\}, \quad s \in \mathcal{S}_\xi, \quad i \in \mathbb{Z},$$

where Φ_ξ is the flow of ξ and ℓ_s the transversal associated to s in \mathcal{T}_ξ , is a locally finite open cover of \mathbb{R}^2 . Indeed by hypothesis the union of the $\pi_\xi(\ell_i)$ covers \mathcal{F}_ξ and therefore under the flow Φ_ξ the ℓ_i cover the whole plane. Moreover, since the boundary of every canonical region has only finitely many components, only finitely many of the $V_{s,i}$ cover any given point.

Inside each $V_{s,i}$ every point p can be written as $\Phi_\xi^t(q)$ for some $q \in \ell_s$ so that we can define a smooth function $f_{s,i}(\Phi_\xi^t(q)) = \phi(t)$, where ϕ is any smooth function strictly monotonic for $t \in (0, 1)$ and such that $\phi|_{(-\infty, 0)} \equiv 0$ and $\phi|_{(1, \infty)} \equiv 1$. Since each $V_{s,i}$ divides the plane in two connected components, each $f_{s,i}$ can be extended to a smooth function on the whole plane by setting it identically to 1 in the component containing $\Phi_\xi^1(\ell_s)$ and identically 0 in the other. A direct calculation shows that $L_\xi f_{s,i}(p) = \phi'(t) > 0$ within each $V_{s,i}$ while $L_\xi f_{s,i}$ is identically zero outside of it. Now recall that the set $\mathcal{S}_\xi \times \mathbb{Z}$ is countable and let $n_{s,i}$ be any bijection of it with \mathbb{N} . The series

$$f = \sum_{s \in \mathcal{S}_\xi, i \in \mathbb{Z}} 2^{-n_{s,i}} f_{s,i}$$

converges to a continuous function (because the $f_{s,i}$ are uniformly bounded) which is actually smooth because the derivatives of all positive orders of the $f_{s,i}$ have compact support. By construction $L_\xi f \geq 0$ but the inequality is strict because for every x_0 there exists at least one index (s_0, i_0) such that $L_\xi f_{s_0, i_0} > 0$. \square

Note that the inequality $L_\xi f > \epsilon$, with $\epsilon > 0$, requires stricter conditions to be solvable no matter how small ϵ is. E.g. it admits no smooth solutions if ξ is complete for in that case, as pointed out in the previous section, all gaps of the constant function ϵ (and, a fortiori, all gaps of every function not smaller than it) would be infinite.

4 $L_\xi(C^\infty(\mathbb{R}^2))$

From this point on we will assume that ξ is transversally Hamiltonian and we will denote by $F \in C^\infty(\mathbb{R}^2)$ a generator of $\ker L_\xi$, so that $\ker L_\xi = F^*(C^\infty(\mathbb{R}))$,

by \mathcal{G} the Hamiltonian foliation transversal to \mathcal{F}_ξ and by G any Hamiltonian of \mathcal{G} .

A fundamental tool in our analysis will be the map $\Phi_{FG} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $x' = F(x, y)$, $y' = G(x, y)$. Assume first that ξ is intrinsically Hamiltonian, so that F is regular. In this case Φ_{FG} is an immersion, since also G is regular and the level sets of F and G are everywhere transversal by hypothesis, so that it induces on the source space the following metric and symplectic structures:

$$g_{FG} = \Phi_{FG}^*((dx')^2 + (dy')^2) = (dF)^2 + (dG)^2, \quad \Omega_{FG} = \Phi_{FG}^*(dx' \wedge dy') = dF \wedge dG.$$

In particular Φ_{FG} induces on the source space complex structure J_{FG} , whose real and imaginary spaces are $T\mathcal{F}_\xi$ and $T\mathcal{G}$, and a Poisson structure $\{\cdot, \cdot\}_{FG}$. Via Ω_{FG} we can build a pair of commuting regular vector fields respectively tangent to \mathcal{F}_ξ and \mathcal{G} . Recall that the Hamiltonian vector field η associated to a Hamiltonian H with respect to a symplectic form Ω is defined by the relation $i_\eta \Omega = dH$; in this case we write, with a slight abuse of notation, that $\eta = \Omega^{-1}(dH)$.

Proposition 4. *Let $\xi'_F = -\Omega_{FG}^{-1}(dF)$, $\xi_F = -\Omega_0^{-1}(dF)$, $\xi'_G = \Omega_{FG}^{-1}(dG)$ and $\xi_G = \Omega_0^{-1}(dG)$. The following relations hold:*

1. $\Omega_{FG} = (L_{\xi_F} G) \Omega_0$.
2. $\xi'_F = \frac{1}{L_{\xi_F} G} \xi_F$, $\xi'_G = \frac{1}{L_{\xi_F} G} \xi_G$.
3. $L_{\xi'_F} F = 0$, $L_{\xi'_F} G = 1$, $L_{\xi'_G} F = 1$, $L_{\xi'_G} G = 0$.
4. $(\Phi_{FG})_*(\xi'_F) = \partial_{y'}$ and $(\Phi_{FG})_*(\xi'_G) = \partial_{x'}$ within $\Phi_{FG}(\mathbb{R}^2)$.
5. $\{F, G\}_{FG} = L_{\xi_F} G = 1$.
6. $[\xi'_F, \xi'_G] = 0$.
7. The pair (ξ'_F, ξ'_G) is an orthonormal base for g_{FG} .
8. $L_{\xi'_F} g_{FG} = L_{\eta'} g_{FG} = 0$.
9. $L_{\xi'_F} \Omega_{FG} = L_{\eta'} \Omega_{FG} = 0$.
10. $J_{FG} \xi'_F = \xi'_G$, $J_{FG} \xi'_G = -\xi'_F$.

Proof. 1. A direct calculation show that $\xi_F = -\partial_y F \partial_x + \partial_x F \partial_y$, so that $dF \wedge dG = (\partial_x F \partial_y G - \partial_y F \partial_x G) dx \wedge dy = (L_{\xi_F} G) \Omega_0$.

2. It is a direct consequence of the definition of ξ'_F and ξ'_G and (1).

3. It is a direct consequence of (2).

4. Since Φ_{FG} is not an injection, the push-forward of a vector field $(\Phi_{FG})_*(\zeta) = T\Phi_{FG} \circ \zeta \circ \Phi_{FG}^{-1}$ is not well-defined unless $T\Phi_{FG}(\zeta)$ takes the same value

on all points of $\Phi_{FG}^{-1}(p)$ for every $p \in \Phi_{FG}(\mathbb{R}^2)$. This is the case for ξ'_F and ξ'_G since we get in both cases a constant vector field:

$$((\Phi_{FG})_*(\xi'_F))(x') = \xi'_F(\Phi_{FG}^*(x')) = \xi'_F(F) = 0$$

$$((\Phi_{FG})_*(\xi'_G))(y') = \xi'_G(\Phi_{FG}^*(y')) = \xi'_G(G) = 1$$

and similarly for ξ'_G .

5. $\{F, G\}_{FG} = \{\Phi_{FG}^* x', \Phi_{FG}^* y'\}_{FG} = \Phi_{FG}^* \{x', y'\}_0 = \Phi_{FG}^* 1 = 1$.
6. $[\xi'_F, \xi'_G] = [-\Omega_{FG}^{-1}(dF), \Omega_{FG}^{-1}(dG)] = \Omega_{FG}^{-1}(\{F, G\}_{FG}) = \Omega_{FG}^{-1}(1) = 0$.
7. $g_{FG}(\xi'_F, \xi'_F) = (dF(\xi'_F))^2 + (dG(\xi'_F))^2 = (L_{\xi'_F} F)^2 + (L_{\xi'_F} G)^2 = 0 + 1$ and similarly for the other combinations.
8. It is a direct consequence of the previous item.
9. This just restates that ξ'_F and ξ'_G are Hamiltonian with respect to Ω_{FG} .
10. It is due to the fact that both g_{FG} and Ω_{FG} are in canonical form with respect to ξ'_F and ξ'_G .

□

When ξ is not intrinsically Hamiltonian but is of finite type then F is not globally regular but nevertheless its differential goes to zero only on some of the separatrices, so that the restriction of Φ_{FG} to each of the canonical regions of ξ is still an immersion. Correspondingly, the pair of commuting regular vector fields ξ'_F and ξ'_G is well defined within the canonical regions but, while ξ'_F is globally well-defined, ξ'_G diverges on the separatrices where dF is zero. Note that there is no way to find a global substitute for ξ'_G :

Proposition 5. *Let \mathcal{F} be a plane foliation of finite type. Then a pair of commuting regular linearly independent vector fields (ξ, η) , with ξ tangent to \mathcal{F} , exists iff \mathcal{F} is Hamiltonian.*

Proof. We showed in previous proposition that such pair always exists if \mathcal{F} is Hamiltonian. Assume then that it is not. In this case we can always find a smooth function F with no maxima or minima whose differential vanishes on some of the separatrices and whose level sets are the leaves of \mathcal{F} and a second function G , this one regular on the whole plane, whose level sets are always transversal to \mathcal{F} . Correspondingly we can always find two vector fields ξ and η s.t.

$$L_\xi F = 0, \quad L_\xi G = 1, \quad L_\eta G = 0, \quad L_\eta F \geq 0.$$

Let now α e β the two smooth functions s.t. $[\xi, \eta] = \alpha\xi + \beta\eta$. Then

$$\alpha = \alpha L_\xi G + \beta L_\eta G = L_{[\xi, \eta]} G = L_\xi(L_\eta G) - L_\eta(L_\xi G) = 0$$

while

$$\beta L_\eta F = \alpha L_\xi F + \beta L_\eta F = L_{[\xi, \eta]} F = L_\xi(L_\eta F) - L_\eta(L_\xi F) = L_\xi(L_\eta F)$$

namely $\beta = L_\xi[\log L_\eta F]$. Since $[\xi, \eta]$ has only the η component, the only thing we can do to make the commutator vanish is multiplying η by some non-zero factor μ since any other change would just introduce a ξ component. On the other side

$$[\xi, \mu \eta] = L_\xi \mu \eta + \mu [\xi, \eta] = L_\xi \mu \eta + \mu \beta \eta$$

leading to $\mu = 1/L_\eta F$; this function though is not smooth because the differential of F vanishes on some of the separatrices. \square

Let us turn now to the study of the image of L_ξ . This is clearly equivalent to studying the image of $L_{\xi'_F}$ but the latter is more convenient for the following two propositions:

Proposition 6. *The cohomological equation $L_{\xi'_F} f(x, y) = g(x, y)$, restricted to the subalgebra $\Phi_{FG}^*(C^\infty(\mathbb{R}^2)) = \{\Phi_{FG}^* \hat{f} \mid \hat{f} \in C^\infty(\mathbb{R}^2)\}$, writes, in the image of Φ_{FG} , as*

$$\frac{\partial}{\partial y'} \hat{f}(x', y') = \hat{g}(x', y') \quad (2)$$

where $\hat{f} = (\Phi_{FG})_* f$ and $\hat{g} = (\Phi_{FG})_* g$.

Proof. In general Φ_{FG} is not injective so that, while the pull-back of function $\Phi_{FG}^* \hat{f} := \hat{f} \circ \Phi_{FG}$ is well-defined on the whole $C^\infty(\mathbb{R}^2)$, the push forward $(\Phi_{FG})_* f := f \circ \Phi_{FG}^{-1}$ leads to a well-defined function only within the subalgebra $\Phi_{FG}^*(C^\infty(\mathbb{R}^2))$. Then from point (3) of Proposition 4 follows that

$$(\Phi_{FG})_* \left(L_{\xi'_F} \left(\Phi_{FG}^* \hat{f} \right) \right) = L_{(\Phi_{FG})_* \xi'_F} \left((\Phi_{FG})_* \Phi_{FG}^* \hat{f} \right) = \frac{\partial}{\partial y'} \hat{f}$$

\square

Theorem 3. *A function $g \in C^\infty(\mathbb{R}^2)$ belongs to $L_{\xi'_F}(C^\infty(\mathbb{R}^2))$ iff all functions $L_{\xi'_G}^k g$, $k \in \mathbb{N}$, have finite gap for all pairs of adjacent separatrices of ξ'_F .*

Proof. As we already pointed out, every continuous solution to $L_{\xi'_F} f = g$ is automatically smooth in the ξ'_F direction, i.e. $L_{\xi'_F}^k f$ is continuous for every $k \in \mathbb{N}$.

Assume first that ξ'_F is intrinsically Hamiltonian. Since ξ'_F and ξ'_G commute and are globally well-defined, the first derivative in the ξ'_G direction satisfies the cohomological equation $L_{\xi'_F} (L_{\xi'_G} f) = L_{\xi'_G} g$ and analogously the k -th derivative in the ξ'_G direction satisfies $L_{\xi'_F} (L_{\xi'_G}^k f) = L_{\xi'_G}^k g$. Now we can use the claim of Lemma 2 to conclude that each $L_{\xi'_G}^k f$ is globally continuous iff $L_{\xi'_G}^k g$ has finite gap between every pair of adjacent separatrices.

Assume now that ξ'_F is of finite type, so that ξ'_G is only well-defined within the canonical regions of ξ'_F . By repeating the same kind of arguments used in Lemma 2 it is clear that we can extend a smooth solution within a saturated

open set to the whole plane iff the gap of $L_{\xi'_G}^k g$ has finite gap between every pair of adjacent separatrices. Note indeed that in the definition of gap the values of ξ'_G on the separatrices are never used so the fact that ξ'_G diverges on some of them does not hinder the evaluation of the gap. \square

From Proposition 6 and the surjectivity of $\partial_{y'}$ we get a large explicit subalgebra of the image of $L_{\xi'_F}$:

Proposition 7. $\Phi_{FG}^*(C^\infty(\mathbb{R}^2)) \subset L_{\xi'_F}(C^\infty(\mathbb{R}^2))$

This fact corresponds to two elementary observations: one, algebraic, that

$$L_{\xi'_F} \hat{f}(F, G) = L_{\xi'_F} F \partial_{x'} \hat{f}(F, G) + L_{\xi'_F} G \partial_{y'} \hat{f}(F, G) = \partial_{y'} \hat{f}(F, G);$$

the other, geometric, that the constant vertical vector field $\partial_{y'}$ on $\Phi_{FG}(\mathbb{R}^2)$ can always be extended to the whole plane, where it is surjective on $C^\infty(\mathbb{R}^2)$.

5 Local behaviour of functions of $L_\xi(C^\infty(\mathbb{R}^2))$ close to a pair of adjacent separatrices

Proposition 6 shows that locally, in the image of the map Φ_{FG} , the cohomological equations relative to vector fields ξ'_F look all the same, independently on the topology of their leaf spaces; the qualitative difference between them resides rather in the global geometry of the map Φ_{FG} . It is easy to verify that, as soon as ξ'_F has at least two pairs of separatrices, Φ_{FG} cannot be injective, which is not optimal for several reasons. We bypass this problem by considering the map $\hat{\Phi}_{FG} : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ defined by $\hat{\Phi}_{FG}(x, y) = (x, y, F(x, y), G(x, y))$. By construction $\hat{\Phi}_{FG}$ is a diffeomorphism between \mathbb{R}^2 and $\Gamma_{FG} = \hat{\Phi}_{FG}(\mathbb{R}^2) \subset \mathbb{R}^4$, the graph of Φ_{FG} . The symplectic, metric and almost complex structures determined on \mathbb{R}^2 by F and G , as pointed out at the beginning of the previous section, induce the same structures on Γ_{FG} via the push-forward $(\hat{\Phi}_{FG})_*$. We use on $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ coordinates (x, y, x', y') and denote by π_1 and π_2 the projections on the first and second factor. By definition $\pi_1 \circ \hat{\Phi}_{FG} = \text{id}_{\mathbb{R}^2}$ and $\pi_2 \circ \hat{\Phi}_{FG} = \Phi_{FG}$, so Γ_{FG} admits (x, y) as global coordinates and (F, G) as local coordinates at every point. A direct calculation shows that

$$(\hat{\Phi}_{FG})_*(\xi'_F) = \xi'_F \oplus \partial_{y'}, \quad (\hat{\Phi}_{FG})_*(\xi'_G) = \xi'_G \oplus \partial_{x'}.$$

In particular the projection on the second factor of the image of the leaves of \mathcal{F}_ξ and \mathcal{G} are, respectively, vertical and horizontal straight lines in the plane (x', y') . All leaves which are inseparable one from the other are mapped to disjoint open intervals of the same line, so that the images in the graph of any pair of adjacent separatrices of ξ'_F are separated by a vertical closed bounded interval I .

Proposition 8. *For every pair of separatrices s_1 and s_2 of ξ'_F , with $a = F|_{s_1 \cup s_2}$, there exists a saturated open neighbourhood U of s_1 and s_2 on which Φ_{FG} is injective and $\Phi_{FG}(U \cap \Phi_{FG}^{-1}((a_1, a_2) \times (c_1, c_2))) = (a_1, a_2) \times (c_1, c_2) \setminus R$, where $R = [a, a_2] \times [b_1, b_2]$ or $R = (a_1, a] \times [b_1, b_2]$, both a_i and c_i can be infinite and $c_1 < b_1 \leq b_2 < c_2$.*

Proof. Let $p_i \in s_i$, $i = 1, 2$, be any two points on the two separatrices, set $c_i = G(p_i)$ and denote by ℓ_i be the two leaves of \mathcal{G} passing through the p_i . The two numbers c_1 and c_2 cannot be equal since the restriction of G to any leaf of $\mathcal{F}_{\xi'_F}$ is strictly monotonic and, because of the inseparability of s_1 and s_2 , there are leaves of $\mathcal{F}_{\xi'_F}$ cutting both ℓ_1 and ℓ_2 ; in particular $G(s_1) \cap G(s_2) = \emptyset$. Assume that $c_1 < c_2$ (otherwise switch the names of the points), set $U_i = \pi_{\xi}^{-1}(\ell_i)$, $i = 1, 2$ and denote by V and Λ respectively the union and intersection of U_1 and U_2 .

Assume first that Λ is contained in $F < a$. We claim that the restriction of Φ_{FG} to V is injective. Indeed let $A_i = U_i \setminus \Lambda$, $i = 1, 2$, so that $V = \Lambda \sqcup A_1 \sqcup A_2$. Clearly $\Phi_{FG}|_{\Lambda}$ is injective since Λ fibers on $\ell_1 \cap \Lambda$, each fiber being a leaf of cF_{ξ} , with G strictly monotonic on each fiber and F strictly monotonic on the base. Moreover, $F(\Lambda) \subset (-\infty, a)$ by assumption. Similarly, each A_i fibers on $\ell_i \cap A_i$ so that $\Phi|_{A_i}$ is injective too; this time though $F(A_i) \subset [a, \infty)$ and, moreover, $G(A_i) = G(s_i)$. Consider now the set $V' = V \cap G^{-1}((c_1, c_2))$ and let s be any leaf of \mathcal{F}_{ξ} inside Λ . The sets of leaves of $\mathcal{G}|_{V'}$ intersecting, respectively, s_1 and s_2 cut s in two disjoint open intervals (c_1, b_1) and (b_2, c_2) ; in particular all leaves of $\mathcal{G}|_{V'}$ corresponding to the values in the closed interval $[b_1, b_2]$ do not intersect neither s_1 nor s_2 and are such that s_1 and s_2 lie on different components with respect to each of them. Finally, let $F(\ell_1) = (a_1, a'_2)$ and $F(\ell_2) = (a_1, a''_2)$. Then $\Phi_{FG}(\Lambda \cap G^{-1}((c_1, c_2))) = (a_1, a) \times (c_1, c_2)$, $\Phi_{FG}(A_1 \cap G^{-1}((c_1, c_2))) = [a, a'_2] \times (c_1, b_1)$ and $\Phi_{FG}(A_2 \cap G^{-1}((c_1, c_2))) = [a, a''_2] \times (b_2, c_2)$ so that $\Phi_{FG}(V \cap F^{-1}((a_1, a_2)) \cap G^{-1}((c_1, c_2))) = (a_1, a_2) \times (c_1, c_2) \setminus R$ for $a_2 = \min\{a'_2, a''_2\}$.

In case V is contained in $F > a$, we use the chart $\tilde{\Phi}_{FG} = (-F, G)$ and repeat the argument above. \square

We call the chart $(U \cap \Phi_{FG}^{-1}((a_1, a_2) \times (c_1, c_2)), \Phi_{FG})$ ⁵ a *normal chart* for s_1 and s_2 . By Proposition 3 there are countably many conditions that must be satisfied for each one of the intervals between pairs of adjacent separatrices so that equation (2) admits a smooth solution. Since in Γ_{FG} there is a natural family of transversals for \mathcal{F}_{ξ} these conditions can be restated more properly for this setting in the following way. Let $I = \{a\} \times [b_1, b_2]$ the vertical interval separating a pair of adjacent separatrices s_1 and s_2 in a normal chart. Every such interval determines a rings homomorphism $\theta_I^{(r)} : SG_I^r \rightarrow SG_a^r$ defined as follows. Given $\mathfrak{g} \in SG_I^r$, let $\hat{g} \in \mathfrak{g}$ and $\delta = \min\{c_2 - b_2, b_1 - c_1\}$, choose an arbitrary $\epsilon \in (0, \delta)$ and set $h_I(x') = \int_{b_1-\epsilon}^{b_2+\epsilon} \hat{g}(x', y') dy'$ for $x' \in (a_1, a)$; we define $\theta_I^{(r)}(\mathfrak{g}) = [h_I]_{SG_a^r}$.

⁵Replace Φ_{FG} with $\tilde{\Phi}_{FG}$ if, in the terminology of Proposition 8, V is contained in $F > a$.

Proposition 9. *The left singular germ of h_I , modulo germs of smooth functions, does not depend on the particular choice of $\epsilon \in (0, \delta)$ and $\hat{g} \in \mathfrak{g}$.*

Proof. Let $h'_I(x') = \int_{b_1-\epsilon'}^{b_2+\epsilon'} \hat{g}'(x', y') dy'$ for $\epsilon' \in (0, \delta)$ and $\hat{g}' \in \mathfrak{g}$. Then the function

$$\begin{aligned} h'_I(x') - h_I(x') &= \int_{b_1-\epsilon'}^{b_2+\epsilon'} (\hat{g}'(x', y') - \hat{g}(x', y')) dy' + \\ &+ \int_{b_2+\epsilon}^{b_2+\epsilon'} \hat{g}(x', y') dy' + \int_{b_1-\epsilon'}^{b_1-\epsilon} \hat{g}(x', y') dy' \end{aligned}$$

is smooth in $(a_1, a]$ since the integrands are all smooth in R_I , the last two because g is smooth in $R_I \setminus I$ and the integral intervals lie inside that set for every $x \in (a_1, a]$ and the first because by hypothesis $\hat{g}' - \hat{g}$ is identically zero in some left neighbourhood of I . Adding to g and \hat{g} any function smooth in the whole R_I changes the rhs just by a smooth function. \square

The maps $\theta_I^{(r)}$ then are well-defined. It is clear from the definition of h_I that $\theta_I^{(r)}$ is a $C_x^r(\mathbb{R})$ -module homomorphism, where $C_x^r(\mathbb{R})$ is the algebra of C^r functions depending on x' only, since

$$\int_{b_1-\epsilon}^{b_2+\epsilon} f(x') \hat{g}(x', y') dy' = f(x') \int_{b_1-\epsilon}^{b_2+\epsilon} \hat{g}(x', y') dy',$$

and commutes with the derivatives with respect to x' , i.e. $\theta_I^{(r)}(\partial_{x'}^k \hat{g}) = \partial_{x'}^k \theta_I^{(r)}(\hat{g})$.

Next proposition shows that the maps $\theta_I^{(r)}$ determine the solvability of the cohomological equation.

Theorem 4. *Let $\{I_j\}$ be the set of all (vertical, closed) intervals between adjacent separatrices in Γ_{FG} and $\theta_{I_j}^{(r)}$ the corresponding ring homomorphisms. Then $g \in L_{\xi'_F}(C^r(\mathbb{R}^2))$ iff $[(\hat{\Phi}_{FG})_* g]_{SG_{I_j}^r} \in \ker \theta_{I_j}^{(r)}$ for all $\theta_{I_j}^{(r)}$.*

Proof. Let $I = \{a\} \times [b_1, b_2]$ be the vertical interval which separates two adjacent separatrices of ξ'_F in a normal chart for the corresponding adjacent separatrices s_1 and s_2 and set $\hat{g} = (\hat{\Phi}_{FG})_* g$ within that chart. Then

$$\lim_{x' \rightarrow a^-} \int_{b_1-\epsilon}^{b_2+\epsilon} \partial_{x'}^k \hat{g}(x', y') dy'$$

is exactly the gap of $\Phi_{FG}^* g$ between s_1 and s_2 with respect to the pair of transversals which are the counterimages of $y' = b_1 - \epsilon$ and $y' = b_2 + \epsilon$ and the gap exists and is finite if and only if those functions can all be extended to continuous functions for all $k < r$, which in turn means that the (germ of the) function $\int_{b_1-\epsilon}^{b_2+\epsilon} \partial_{x'}^k \hat{g}(x', y') dy'$ can be extended to a smooth map up to $x' = a$, i.e. $[(\hat{\Phi}_{FG})_* g]_{SG_I^r} \in \ker \theta_I^{(r)}$. Now the claim follows immediately from Theorem 3. \square

The $C^r(\mathbb{R})$ -modules $\Theta_{I_j}^r = \ker \theta_{I_j}^{(r)}$ contain therefore the (left singular) germs of all functions for which the cohomological equation is solvable in the neighbourhood of a pair of adjacent separatrices. Modulo isomorphisms there are only two such spaces: the one relative to $J = \{0\} \times [-1, 1]$ and the one relative to $O = \{(0, 0)\}$; moreover $\Theta_O^r \subset \Theta_J^r$.

Proposition 10. *The spaces Θ_O^r satisfy the following properties:*

1. Θ_O^r contains the singular left germs of all y' -odd⁶ C^r functions;
2. Θ_O^r contains the singular left germs of some but not all y' -even C^r functions;
3. Θ_O^{r+1} is strictly contained in Θ_O^r .

Proof. 1. If \hat{g} is y' -odd then also every $\partial_{x'}^k \hat{g}$ is so for every $k \leq r$; then $\int_{-\epsilon}^{\epsilon} \partial_{x'}^k \hat{g}(x', y') dy'$ is identically zero for every $k \leq r$ and therefore it can be extended smoothly to a C^r function up to $x' = 0$.

2. Consider $\hat{g}(x', y') = e^{-(y')^2/(x')^2} / \sqrt{-\pi x'} \in C^\infty(\mathbb{R}^2 \setminus (0, 0))$, so that

$$\lim_{x \rightarrow 0^-} g(x', y') = 0, \quad y' \neq 0; \quad \lim_{x \rightarrow 0^-} g(x', 0) = \infty; \quad \int_{-\infty}^{\infty} g(x', y') dy' = 1, \quad \forall x' \in \mathbb{R}.$$

By reparametrizing the y' coordinate we can find a \hat{g}' with the same limits with respect to $x' \rightarrow 0$ but such that $\int_{-\epsilon}^{\epsilon} \hat{g}'(x', y') dy' = 1$. Since the $\theta_O^{(r)}$ are homomorphisms of $C_x^r(\mathbb{R})$ -modules we can get in this way every C^r function $f(x')$ just by multiplying $\hat{g}'(x', y')$ by $f(x')$. On the other side, germs of functions diverging too fast, e.g. as $\hat{g}(x', y') = (x')^{-2} + (y')^{-2}$, do not belong to any Θ_O^r .

3. Consider $\hat{g}(x', y') = \frac{x'}{\sqrt{(x')^2 + (y')^2}} \in C^\infty(\mathbb{R}^2 \setminus (0, 0))$. The germ of the corresponding $h_O(x') = 2x' \log \left[2 \left(y' + \sqrt{(x')^2 + (y')^2} \right) \right]_{y'=0}^{\epsilon}$ can be extended at 0 to a C^0 (but not C^1) function. By integrating r times \hat{g} with respect to x' one can get concrete examples of functions smooth in $\mathbb{R}^2 \setminus (0, 0)$ whose germ belongs to Θ_O^r but not to Θ_O^{r+1} . \square

An immediate consequence of point (3) of the proposition above is the following:

Corollary 1. *Let $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$, $L_\xi^{(r)}$ the restriction of L_ξ to $C^r(\mathbb{R}^2)$ and let $C_\xi^r(\mathbb{R}^2)$ be the set of all functions $f \in C^r(\mathbb{R}^2)$ such that $f + g$ is at most C^r for all $g \in \ker L_\xi^{(r)}$. The inclusions*

$$L_\xi^{(r+1)} \left(C_\xi^{r+1}(\mathbb{R}^2) \right) \cap C^\infty(\mathbb{R}^2) \subset L_\xi^{(r)} \left(C_\xi^r(\mathbb{R}^2) \right) \cap C^\infty(\mathbb{R}^2)$$

are strict for every $r \in \mathbb{N}$.

⁶We say that $f(x, y)$ is y -odd if $f(x, -y) = -f(x, y)$ and y -even if $f(x, -y) = f(x, y)$.

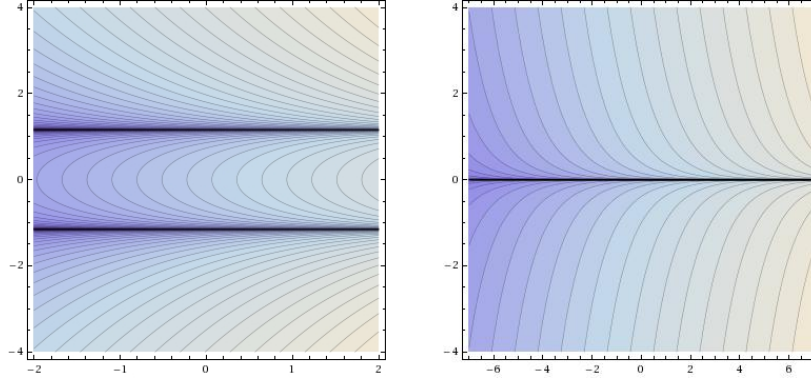


Figure 1: Level sets of $F(x, y) = (y^2 - 1)e^x$ (left) and $G(x, y) = ye^x$ (right). The first foliation has separatrices $y = \pm 1$, the second has none.

Proof. The fact that $L_\xi^{(r+1)}(C^{r+1}(\mathbb{R}^2)) \cap C^\infty(\mathbb{R}^2) \subset L_\xi^{(r)}(C^r(\mathbb{R}^2)) \cap C^\infty(\mathbb{R}^2)$ is trivially true because $L_\xi^{(r)}(f + g) \in C^\infty(\mathbb{R}^2)$ for each $f \in C^\infty(\mathbb{R}^2)$, $g \in \ker L_\xi^{(r)}$. Our claim is that the inclusion is true even when we restrict L_ξ to the space of functions which are “strongly C^r ” with respect to ξ , i.e. those that cannot be made smoother by adding to them an element of the kernel of $L_\xi^{(r)}$. Consider indeed the concrete case used in point (3) of Proposition 10: in a normal chart, where the two separatrices are given by $x' = 0$, $y' > a$ and $x' = 0$, $y' < a$, the (local) primitive of $\hat{g}(x', y') = x' / \sqrt{(x')^2 + (y')^2}$ is $f(x', y') = x' \log \left[2 \left(y' + \sqrt{(x')^2 + (y')^2} \right) \right]$, which is C^0 but not C^1 because the first derivative with respect to x' diverges on the second separatrix. Since the divergence takes place only on one of the separatrices, there is no way to eliminate it by adding a function belonging to the kernel of L_ξ . \square

In the following subsections we work out in detail two model examples.

5.1 $\xi_n = (1 - n + (1 + n)y) \partial_x + (1 - y^2) \partial_y$

The ξ_n , $n \in \mathbb{N}$, are all of finite type since they are polynomial. In particular they all have exactly two separatrices, the straight lines $y = \pm 1$, which bound the canonical region $\mathbb{R} \times (-1, 1)$. The function $F_n(x, y) = (1 + y)^n(1 - y)e^x$ is a functional generator for $\ker L_{\xi_n}$ so the only intrinsically Hamiltonian among them is $\xi_1 = 2y \partial_x + (1 - y^2) \partial_y$. All of them are transversal to the same Hamiltonian foliation \mathcal{G} of the level sets of $G(x, y) = ye^x$, which is topologically conjugate with the trivial foliation in parallel straight lines. The 2-form

$$\Omega_{FG} = 2(1 + y)^{n-1}(1 - (n - 1)y + ny^2)e^{2x} \Omega_0$$

is degenerate on the separatrix $y = -1$ except in the $n = 1$ case, when is globally non-degenerate. Via Ω_{FG} we get

$$\xi'_{F_n} = \frac{1}{2e^x(1 - (n-1)y + ny^2)} \xi_n, \quad \xi'_{G_n} = \frac{1}{2e^x(1+y)^{n-1}(1 - (n-1)y + ny^2)} \eta,$$

where $\eta = 2\partial_x - 2y\partial_y$. Due to the degeneracy of Ω_{FG} , ξ'_{G_n} diverges on the separatrix $y = -1$ for $n \neq 1$.

The image of every $\Phi_{F_n G}$ is $\mathbb{R}^2_0 = \mathbb{R}^2 \setminus \{0\} \times [0, \infty)$ and $\Phi_{F_1 G}$ is an almost complex map between (\mathbb{R}^2, J_{FG}) and (\mathbb{R}^2_0, i) for

$$J_{FG} = y\partial_x \otimes dx + 2\partial_x \otimes dy - (1+y^2)/2\partial_y \otimes dx - y\partial_y \otimes dy.$$

The leaves of \mathcal{F}_{F_n} within the canonical region are sent to the vertical lines of the half plane $x < 0$ and the separatrices $y = -1$ and $y = +1$ to the half lines $\{0\} \times (-\infty, 0)$ and $\{0\} \times (0, +\infty)$ respectively. The leaves lying in the half-plane $y > 1$ fill in the vertical half-lines the first quadrant and the ones lying in $y < 1$ the fourth quadrant. In this case the maps $\Phi_{F_n G}$ are all globally injective. The cohomological equation $L_{\xi_{F_n}} f = g$ maps to

$$\partial_{y'} \hat{f}(x', y') = \hat{g}(x', y'), \quad \hat{g} \in C^\infty(\mathbb{R}^2_0). \quad (3)$$

When \hat{g} is smooth on the whole plane clearly (3) is always solvable. E.g. all smooth solutions to

$$L_{\xi'_{F_n}} f(x, y) = F_n(x, y)G(x, y) = 2(y^2 - 1)(y + 1)^{n-1}ye^{2x}$$

are given by

$$f(x, y) = \frac{F_n(x, y)G^2(x, y)}{2} + h(F_n(x, y)) = 2(y^2 - 1)(y + 1)^{n-1}y^2e^{3x} + h(F_n(x, y)),$$

where $h \in C^\infty(\mathbb{R})$.

In the following we assume $n = 1$ since expressions are much simpler in this case. Consider first the y' -odd function

$$\hat{g}(x', y') = \frac{y'}{\sqrt{(x')^2 + (y')^2}} \in C^\infty(\mathbb{R}_0), \quad \Phi_{FG}^* \hat{g}(x, y) = \frac{2y}{1+y^2} \in C^\infty(\mathbb{R}^2).$$

By Proposition 10 the singular left germ of \hat{g} belongs to Θ_O^∞ and therefore $g \in L_\xi(C^\infty(\mathbb{R}^2))$. Indeed (3) in this case is solved by

$$\hat{f}(x', y') = \sqrt{(x')^2 + (y')^2},$$

whose pull-back

$$\Phi_{FG}^* f(x, y) = (1 + y^2)e^x$$

is globally smooth. Similarly, $y \in L_{\xi'}(C^\infty(\mathbb{R}^2))$ since $y = \Phi_{FG}^* \hat{g}(x, y)$ for the y' -odd singular function $\hat{g}(x', y') = (\sqrt{(x')^2 + (y')^2} + x')/y'$.

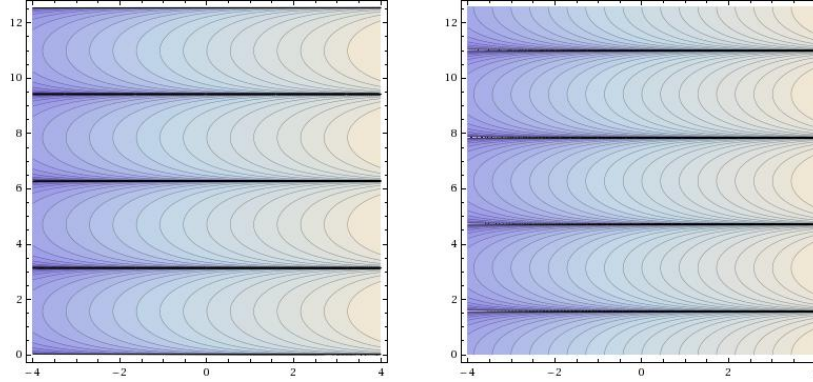


Figure 2: Level sets of $F(x, y) = e^x \sin y$ (left) and $G(x, y) = e^x \cos y$ (right). The separatrices of the first foliation are the straight lines $s_k = \{y = k\pi\}$, $k \in \mathbb{Z}$, the ones of the second are the straight lines $s'_k = \{y = \pi/2 + k\pi\}$, $k \in \mathbb{Z}$. Note that $\mathcal{I}_{s_n} = \{s_{n-1}, s_{n+1}\}$, i.e. s_n is inseparable only from s_{n-1} and s_{n+1} (this is possible because the relation of inseparability is not transitive). The same holds for the s'_k .

On the contrary, in case of

$$\hat{g}(x', y') = \frac{x'}{\sqrt{(x')^2 + (y')^2}}, \quad g(x, y) = \Phi_{FG}^* \hat{g}(x, y) = \frac{1 - y^2}{1 + y^2},$$

as discussed in Proposition 10 we have that the germ of \hat{g} belongs to Θ_O^0 but not to Θ_O^1 ; correspondingly all solutions will be C^0 but not C^1 . E.g. an explicit solution is given by

$$f(x, y) = \Phi_{FG}^* \left(x' \log \left[2 \left(y + \sqrt{(x')^2 + (y')^2} \right) \right] \right) = (1 - y^2) e^x (x + 2 \log |1 + y|).$$

Note that Lie derivatives of f are, as expected, smooth with respect to ξ'_{F_1} direction but singular (on the horizontal straight line $y = -1$) with respect to η . In particular, g belongs to $L_{\xi'_F} (L_{loc}^1(\mathbb{R}^2))$ (where the derivative is intended in the weak sense) but does not belong to any $L_{\xi'_{F_1}} (C^k(\mathbb{R}^2))$, $k > 1$. The same happens in case of $x = \Phi_{FG}^* \hat{g}(x, y)$, where $\hat{g}(x', y') = \log(\sqrt{(x')^2 + (y')^2} + x')/2$. For a thorough discussion about locally integrable solutions of regular polynomial vector fields in the plane depending only on one variable see [DGK10].

5.2 $\xi_n = (\cos y + (n - 1) \cos^2(y/2)) \partial_x - \sin y \partial_y$

The ξ_n , $n \in \mathbb{N}$, are all of finite type for their components are Morse functions depending only on one variable; in this case indeed only the vertical lines can be separatrices and they do not accumulate within any compact set. For every ξ_n the set of separatrices is $\mathcal{S} = \{y = k\pi, k \in \mathbb{Z}\}$. The function $F_n(x, y) = -\sin^{n-1}(y/2) \sin y e^x$ is a functional generator for $\ker L_{\xi_n}$ so

that the only intrinsically Hamiltonian among them is $\xi_1 = \cos y \partial_x - \sin y \partial_y$. A Hamiltonian transversal foliation \mathcal{G}_n for ξ_n is given by the level sets of $G_n(x, y) = \cos y e^{x/n}$. The 2-form

$$\Omega_{FG} = [(n-1)(2 \cos y - \cos(2y)) + 3n + 1] \sin^{n-1}(y/2) e^{(n+1)x/n} \Omega_0 / 4n$$

is degenerate on the separatrices $y = 2k\pi$, except of course in the $n = 1$ case when is globally non-degenerate. Via Ω_{FG} we get

$$\xi'_{Fn} = \frac{2ne^{-x/n}}{n+1+(n-1)(\sin^2 y - \cos y)} \xi_n, \quad \xi'_{Gn} = \frac{-2 \sin^{1-n}(y/2) e^{-x}}{n+1+(n-1)(\sin^2 y - \cos y)} \eta,$$

where $\eta = n \sin y \partial_x + \cos y \partial_y$. Due to the degeneracy of Ω_{FG} , ξ'_{Gn} diverges on the separatrices $y = 2k\pi$, $k \in \mathbb{Z}$, for $n \neq 1$.

The image of every Φ_{FnG} is $\mathbb{R}^2 \setminus \{(0,0)\}$. Note that $\Phi_{F_1G_1}$ is an almost complex map with respect to the almost complex structure

$$J_{F_1G_1} = \partial_y \otimes dx - \partial_x \otimes dy,$$

so that $\Phi_{F_1G_1}$ is actually a holomorphic map; in fact, in complex coordinates, $\Phi_{F_1G_1}(z) = e^{z+i\pi/4}$ and its graph is the Riemann surface of the complex logarithm. The graphs of all other Φ_{FnGn} are diffeomorphic to it.

Consider just the case of the coordinate functions x and y . The first is y' -even since $2x = \Phi_{FG}^* \hat{g}(x, y)$ for $\hat{g}(x', y') = \log[(x')^2 + (y')^2]$. A direct calculation shows that

$$[\theta_n(\hat{g})](x') = 2 \int_0^\epsilon \log((x')^2 + (y')^2) dy' = 4x \tan^{-1}(\epsilon/x) + 2\epsilon(\log(\epsilon^2 + x^2) - 2)$$

which can be continued to a smooth function up to $x' = 0$. Hence $\hat{g} \in \Theta_n^\infty$ for all n and, correspondingly, $x \in L_{\xi'_F}(C^\infty(\mathbb{R}^2))$. An explicit solution is given by

$$f(x, y) = \Phi_{FG}^* [2x' \tan^{-1} \frac{y'}{x'} + \log[(x')^2 + (y')^2] - 2y'] = 2[(x-1) \cos y - y \sin y] e^x.$$

The second is y' -odd since $y = \Phi_{FG}^* \hat{g}(x, y)$ for $\hat{g}(x', y') = \tan^{-1}(x'/y')$. Hence even in this case $\hat{g} \in \Theta_n^\infty$ for all n , i.e. $y \in L_{\xi'_F}(C^\infty(\mathbb{R}^2))$. An explicit solution is given by

$$f(x, y) = \Phi_{FG}^* [y' \tan^{-1} \frac{x'}{y'} + \frac{1}{2} \log[(x')^2 + (y')^2]] = -[y \cos y + x \sin y] e^x.$$

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